Modal Analysis of Rotating Flexible Structures

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A discussion of the modal analysis of a rotating flexible structure is presented. Specifically this work is concerned with the free vibration analysis of a rotating cantilever structure. A brief discussion of the formulation of the equations of motion for the system is included to aid in understanding the origin of the various terms appearing in these equations. Finite element techniques and their corresponding matrix form have been assumed to represent the flexible structure. The results are presented as an extension of existing finite element structural analysis procedures to describe the dynamic characteristics of a spinning structure. Illustrative modal analysis results are presented indicating the influence of structural orientation and magnitude of spin on the modal and stability characteristics of the structure.

I. Introduction

CONVENTIONAL modal analysis techniques are not applicable in the case of an elastic structure spinning at a constant angular velocity. This is of interest because numerous structural configurations such as spinning satellites, rotating shafts, and rotating linkages fall into this category. The analysis of these spinning structures differs from that of stationary structures due to the complexity of the accelerations which act throughout the system. In addition to the accelerations resulting from elastic structural deformations, contributions due to Coriolis and centripetal acceleration may be of significance. Also, the stiffness characteristics of the structure may be modified by the steady state internal loads induced by the centrifugal forces.

The "classic" stiffening effect experienced by a rotating radial beam is discussed in many texts. In Ref. 1 the centrifugal force effects are retained in the modal analysis. Early interest in this problem is evidenced by the design charts detailed in Ref. 2. In Ref. 3 an extensive discussion on the modal analysis problem for an elastic structure attached to a rotating base is presented. Included in this reference are comments on the relative merits of various mathematical models of the elastic structure and the modal analysis results for a discrete spring-mass system having specific orientations with respect to the spin axis. The influence of spin on the natural frequencies and mode shapes of a radial beam and the Skylab solar array configuration are discussed in Refs. 4 and 5. In Ref. 6, application of the NASTRAN computer program for the modal analysis of a rotating structure is detailed. The problem of a spinning spacecraft having inertial rotational degrees of freedom in addition to structural flexibility is addressed in Ref. 7. In this reference the eigenvalue problem is formulated in terms of the rotational motion of the structure as a whole and the elastic motion relative to the rotating frame. Of additional interest in this reference is a method of reducing the complex eigenvalue problem to a standard form, namely that of real matrices for both the real and imaginary parts of the eigenvectors.

Discussions concerning the effects of rotational motion on the dynamic analysis of practical spacecraft configurations are presented in Refs. 8 and 9. In Ref. 8 the linearized governing equations of motion are formulated for a flexible structure undergoing arbitrary translations or rotations. An approximate numerical method is then described for obtaining the transfer functions for attitude control system studies. Reference 9 deals with the modal synthesis of an entire spacecraft experiencing constant rotational motion.

This paper presents a discussion of the revisions which must be made to existing finite element structural analysis procedures to allow determination of the modal characteristics of rotating structures. In addition the potential influence of spin on the dynamic characteristics of such a structure is indicated. For these discussions the specific case of a flexible structure attached to a rigid base which is rotating about an axis fixed in inertial space is examined. Typical modal analysis results for a rotating uniform beam are included to illustrate the potential influence of spin and structural arrangement on the modal and stability characteristics of a rotating flexible structure.

II. Problem Formulation

The basic steps in obtaining the governing equations of motion for a rotating flexible structure are outlined to define the sources of the various terms in these equations. This aids in understanding the nature of a number of the problems facing the analyst in solving the corresponding eigenvalue problem. A detailed discussion of this equation of motion formulation is presented in Ref. 10.

As mentioned, the specific structural configuration to be addressed here is that of a flexible structure, an "appendage," attached to a rigid rotating base. The geometric characteristics of such a configuration are illustrated in Fig. 1.

For purposes of these discussions, the position vector $\bar{r}_i$ of point i on the flexible structure is assumed to be composed of two quantities as illustrated in Fig. 1. The first of these, $\bar{p}_i$, is the undeformed position of point i while $\bar{u}_i$ is its translational elastic deformation. Thus, we have the following vector relationship:

$$\bar{r}_i = \bar{p}_i + \bar{u}_i \tag{1}$$

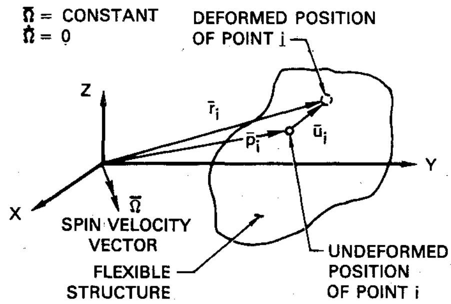

Fig. 1 Deformation of point *i* on flexible structure.

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and the velocity and acceleration of point i are given as

$$\dot{\bar{r}}_i = \bar{\Omega} \times (\bar{p}_i + \bar{u}_i) + \dot{\bar{u}}_i \tag{2}$$

and

$$\ddot{\bar{r}}_i = \bar{\Omega} \times [\bar{\Omega} \times (\bar{p}_i + \bar{u}_i)] + 2\bar{\Omega} \times \dot{\bar{u}}_i + \ddot{\bar{u}}_i \tag{3}$$

In the above it has been assumed that the spin rate is constant, resulting in

$$\dot{\bar{\Omega}} = 0 \tag{4}$$

The form of the position vector $\bar{r}_i$ given by Eq. (1) reflects the assumption that the location and orientation of the spin axis remains fixed in inertial space. Even with this somewhat simplifying assumption, all of the coupling mechanisms between the spin rate and elastic deformation are retained.

Returning to the development of the appendage equations of motion, application of Newton's second law results in the following matrix relationship

$$F = M\ddot{R} \tag{5}$$

where F is the $3I$ by 1 matrix of forces acting at the I control points located throughout the structure. The quantity M is the mass matrix and R is the collection of all the $\ddot{\bar{r}}_i$ accelerations throughout the structure. In addition to externally applied loads, F includes contributions due to structural interactions between neighboring points in the appendage. Thus, combining Eq. (3) with the previous we have

$$M\ddot{U} + 2M\tilde{\Omega}\dot{U} + (K + M\tilde{\Omega}\tilde{\Omega})U = -M\tilde{\Omega}\tilde{\Omega}P \tag{6}$$

In the aforementioned it has been assumed that there are no externally applied loads and F is made up of the structural interaction load K times U where K is the stiffness matrix of the structure and U represents the elastic deformations in the structure. The $3I$ by $3I$ matrix $\tilde{\Omega}$ is the matrix equivalent to the vector cross product and is given as

$$\tilde{\Omega} = \begin{bmatrix} [\ \ ] & & & 0 \\ & \ddots & & \\ & & \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} & \\ & & & \ddots \\ 0 & & & [\ \ ] \end{bmatrix} \tag{7}$$

The term on the right hand side of Eq. (6) defines the centrifugal forces due to the constant spin rate Ω. Thus, the steady state deflected or equilibrium position of the elastic structure due to the presence of Ω may be obtained from

$$[K + M\tilde{\Omega}\tilde{\Omega}]U_{ss} = -M\tilde{\Omega}\tilde{\Omega}P \tag{8}$$

Internal loads will be built up in the structure in going from its undeformed position to the new equilibrium configuration as defined by Eq. (8). These steady-state internal loads of preload effects modifiy the stiffness characteristics of the structure through the introduction of geometric stiffness terms. Several references, such as Przemieniecki in Ref. 11, present techniques for evaluation of the geometric stiffness matrix represented symbolically here as K_G. It should be noted that for large magnitudes of spin rate, determination of the steady state deformations [Eq. (8)] may involve a nonlinear static analysis problem.

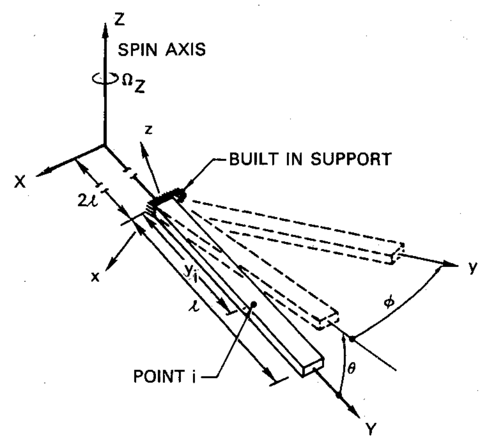

Fig. 2 Uniform beam example.

Incorporating the concept of geometric stiffness terms, Eq. (6) becomes

$$M\ddot{U} + 2M\tilde{\Omega}\dot{U} + (K + K_G + M\tilde{\Omega}\tilde{\Omega})U = 0 \tag{9}$$

It must be remembered that the U's in Eq. (9) now represent elastic deformations measured from the steady-state position defined by Eq. (8). This relationship defines the free vibration or eigenvalue problem which must be addressed to evaluate the influence of spin on the modal characteristics of an elastic structure. The $2M\tilde{\Omega}U$ terms are the Coriolis acceleration contribution, the $M\tilde{\Omega}\tilde{\Omega}U$ terms are the centripetal acceleration contribution, and the $K_G U$ terms are due to the steady-state centrifugal force field.

The system of second order matrix equations detailed by Eq. (9) can be represented by the first-order state equations as

$$\dot{Q} + DQ = 0 \tag{10}$$

where we have

$$Q = \left\{ \begin{array}{c} \dot{U} \\ \hline U \end{array} \right\} \tag{11}$$

and

$$D = \left[\begin{array}{c|c} M^{-1}(2M\tilde{\Omega}) & M^{-1}(K + K_G + M\tilde{\Omega}\tilde{\Omega}) \\ \hline -I & 0 \end{array} \right] \tag{12}$$

In the preceding equation I is the identity matrix and the $3I$ by $3I$ problem of Eq. (9) has become a $6I$ by $6I$ problem. Returning to Eq. (10) we see that the eigenvalue problem is expressed as

$$(\alpha I + D)\phi = 0 \tag{13}$$

where the α's are complex eigenvalues and the ϕ's are the corresponding complex eigenvectors.

III. Illustrative Modal Analysis Results

Modal analysis results for a rotating uniform beam as illustrated in Fig. 2 have been obtained to illustrate the potential influence of spin on the dynamic characteristics of a flexible structure. For this example the spin axis is assumed to be aligned along the Z axis and the components of the spin vector $\tilde{\Omega}$ may be represented in matrix form as

$$\Omega = \begin{Bmatrix} 0 \\ 0 \\ \Omega_Z \end{Bmatrix} \tag{14}$$

The beam, and its coordinate system, are orientated with respect to the spin axis by the two angles θ and ϕ as indicated in Fig. 2. A finite element representation of this beam was assembled and the procedures discussed in the preceding section followed in obtaining modal analysis results.

An eight node finite element representation of the beam was employed for these studies. It was assumed that the beam was made of aluminum with a cross sectional area of 1.6×10^{-3} sq. in. and moments of inertia of 4×10^{-5} (I_x) and 6×10^{-5} (I_z) in.[4] The length of the beam is 40 in.

These selected physical characteristics result in a first-mode frequency ω_{x0} of 16.8 Hz in the $x-y$ plane and a first-mode frequency ω_{z0} of 13.7 Hz in the $y-z$ plane when the spin rate Ω_Z is zero. The following modal analysis results are presented as being modes in either the local $x-y$ or $y-z$ plane. This definition is made in terms of the plane containing the predominant modal deformation for a particular mode.

As illustrated in Figs. 3 and 4 the modal analysis results for a rotating structure are strongly influenced by the orientation of the structure with respect to the spin axis. Results presented in both these figures correspond to a spin rate Ω_Z whose magnitude is one half the nonspin natural frequency ω_{z0}. In these figures the beam natural frequencies are expressed in terms of the ratio of the natural frequency including spin to the corresponding nonspin frequency. As can be seen from these figures, the beam natural frequencies, especially first-mode, may be significantly different than the corresponding nonspin value for certain beam orientations.

To aid in understanding the trends illustrated in Figs. 3 and 4, the behavior of the first mode in each plane was studied in some detail for a number of beam orientations. The trends obtained for one such orientation are shown in Fig. 5. Presented in this figure are the results for increasing spin rate with both θ and ϕ being zero. This orientation corresponds to the classic radial beam configuration. Including the preload effects due to the centrifugal force field results in the well known "stiffening" effect as indicated in Fig. 5. Combining the contribution due to centripetal acceleration with the influence of preload slightly counteracts the stiffening effects of preload in the $x-y$ plane. Including the Coriolis acceleration contribution in addition to the other two effects does not modify the modal results for this particular beam orientation.

The trends given in Fig. 5 may be explained with the aid of equations presented in the Appendix. Referring to Eq. (26) we see that this simplified analysis leads to the following equations of motion for the beam orientation assumed for the results shown in Fig. 5.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_z \end{Bmatrix} + \begin{bmatrix} \omega_{x0}^2 + \Omega_Z^2(\lambda - 1) & 0 \\ 0 & \omega_{z0}^2 + \Omega_Z^2\lambda \end{bmatrix} \begin{Bmatrix} q_x \\ q_z \end{Bmatrix} = 0 \tag{15}$$

As can be seen responses in the $x-y$ and $y-z$ planes are uncoupled. The preload term $\Omega_Z^2\lambda$ increases the frequency in both planes. The parameter λ is defined by Eq. (32). The centripetal acceleration term $-\Omega_Z^2$ appears in only the $x-y$ plane and has a softening effect. The Coriolis acceleration terms do

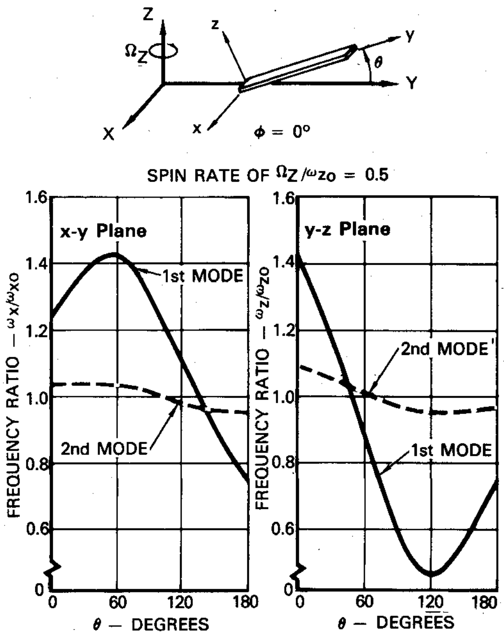

Fig. 3 Influence of beam orientation in the $Y-Z$ plane.

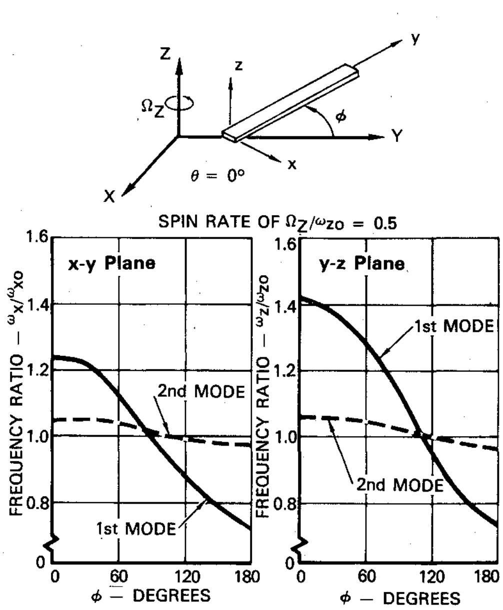

Fig. 4 Influence of beam orientation in the $X-Y$ plane.

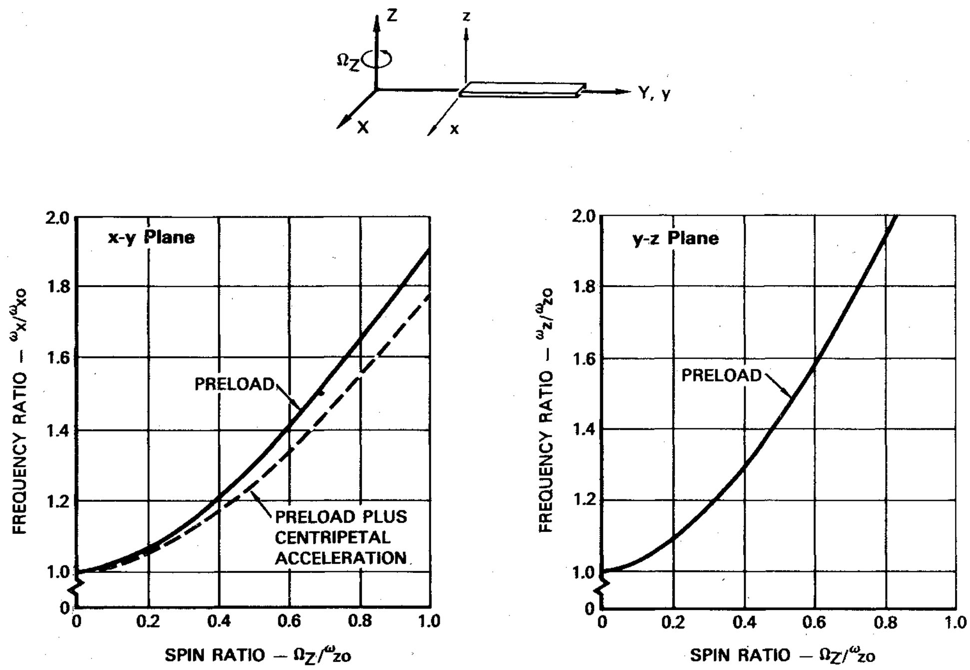

Fig. 5 Comparison of spin-induced effects for $\theta = \phi = 0°$.

not appear for this beam orientation and thus has no influence on the modal analysis results.

Similar results are given in Fig. 6 for a beam orientation defined by a ϕ of zero and a θ of 90°. For this beam orientation the relationships of Eq. (26) become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_x \\ \ddot{q}_z \end{Bmatrix} + \begin{bmatrix} 0 & 2\Omega_Z \\ -2\Omega_Z & 0 \end{bmatrix} \begin{Bmatrix} \dot{q}_x \\ \dot{q}_z \end{Bmatrix} + \begin{bmatrix} \omega_{x0}^2 - \Omega_Z^2 & 0 \\ 0 & \omega_{z0}^2 - \Omega_Z^2 \end{bmatrix} \begin{Bmatrix} q_x \\ q_z \end{Bmatrix} = 0 \tag{16}$$

From Eq. (16) we conclude that preload has no influence on the beam modal frequencies for this particular configuration and the centripetal acceleration term $-\Omega_Z^2$ tends to soften the system in both directions. Both conclusions are supported by the results illustrated in Fig. 6. Also from Fig. 6 we see that when all three spin-induced effects are included, the frequency in the $x-y$ plane increases and that in the $y-z$ plane decreases with increasing rotational speed. As discussed by Likins[10] this spreading of the system frequencies is a characteristic result when the Coriolis acceleration terms are included.

For the particular beam orientation shown in Fig. 6 we see that the first-mode frequency in the $y-z$ plane goes to zero as the spin rate approaches the nonspin frequency ω_{z0}. This zero structural frequency condition corresponds to the situation of an unstable system. Of interest are the stability characteristics of the beam for orientations other than that shown in Fig. 6. Typical stability boundaries for varying beam orientations are presented in Fig. 7. From this figure we see that an unstable system may result for spin rates significantly less than the magnitude of the lowest structural frequency.

The potential influence of spin on the modal response of a structure is illustrated in Fig. 8 where the mode shapes for a particular beam orientation and varying magnitudes of spin rate are presented. These mode shapes are complex in nature indicating a phase relationship between the components of deflection. With no damping in the system, a plus or minus 90° phase relationship results between deflection components in the x and z directions.

IV. Concluding Remarks

A discussion of the modal analysis of a rotating flexible structure has been presented. Finite element structural idealization has been used, thus providing insight into the extension of this structural analysis method to the case of a spinning structure. Illustrative modal analysis results have been presented to show the potential influence of structural orientation and spin rate on the modal and stability characteristics of a structure. As illustrated by these examples, the influence of spin may have a significant effect on the dynamic characteristics of a structure for certain geometric configurations.

A number of points should be considered when conducting eigenvalue solutions of Eq. (9) for large, complex structural configurations. As discussed by Gupta[4-5] a computational technique is available which takes advantage of the banded nature of the matrices associated with the structural idealization. Also a technique is presented in Ref. 7 for converting the complex eigenvalue problem to one defined by real matrices. Both these techniques should be considered when finite element methods are used in dynamic analyses that include the effects of constant rate rotation. In addition, the ramifications of employing Guyan reductions[12] to decrease the potentially large order of Eq. (9) need to be evaluated.

V. Appendix

To aid with the interpretation of the modal analysis results presented in the body of this paper a "two-mode" energy formulation for a spinning beam is presented. The beam is assumed to be uniform, however, it is allowed to be structurally asymmetric. The influences of shear deformation, rotary inertia, and axial deformation are neglected in this development.

Repeating Eq. (2), the velocity of point i on the beam is given as

$$\dot{\bar{r}}_i = \bar{\Omega} \times (\bar{p}_i + \bar{u}_i) + \dot{\bar{u}}_i \tag{17}$$

Referring to Fig. 2, the components of the preceding vector equation, expressed in the beam coordinate system become

$$\dot{r}_x = \cos\phi\dot{u}_x - \Omega_Z \ell_0 - \Omega_Z(\cos\theta\sin\phi u_x + \cos\theta\cos\theta y_i - \sin\phi u_z)$$

$$\dot{r}_y = \cos\theta\sin\phi\dot{u}_x - \sin\theta\dot{u}_z + \Omega_Z(\cos\phi u_x - \sin\phi y_i)$$

$$\dot{r}_z = \sin\theta\sin\phi\dot{u}_x + \cos\theta\dot{u}_x \tag{18}$$

for this uniform beam configuration. It is assumed that the elastic beam deflections in the $y-z$ (Fig. 9) and $x-y$ planes are of the form

$$u_x(y,t) = q_x(t)\left[3\left(\frac{y}{\ell}\right)^2 - \left(\frac{y}{\ell}\right)^3\right] \tag{19}$$

and

$$u_z(y,t) = q_z(t)\left[3\left(\frac{y}{\ell}\right)^2 - \left(\frac{y}{\ell}\right)^3\right] \tag{20}$$

where q_x and q_z are the generalized coordinates of the beam response. The above relationships are based on the static deflection shape of a uniform cantilever beam subjected to a tip load.

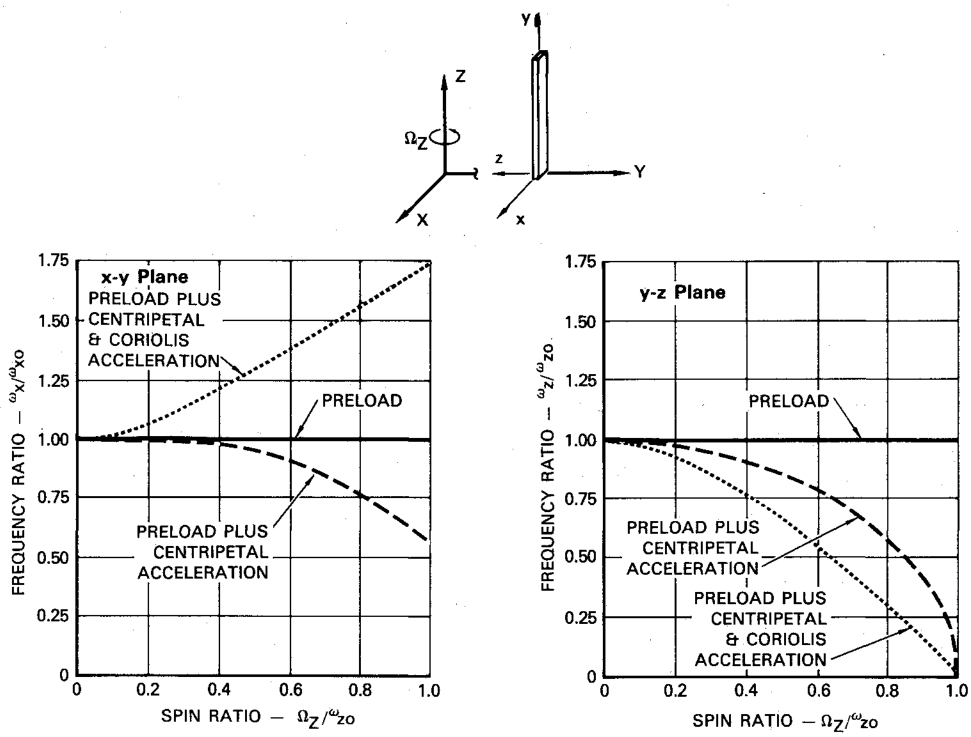

Fig. 6 Comparison of spin-induced effects for $\theta = 90°$ and $\phi = 0°$.

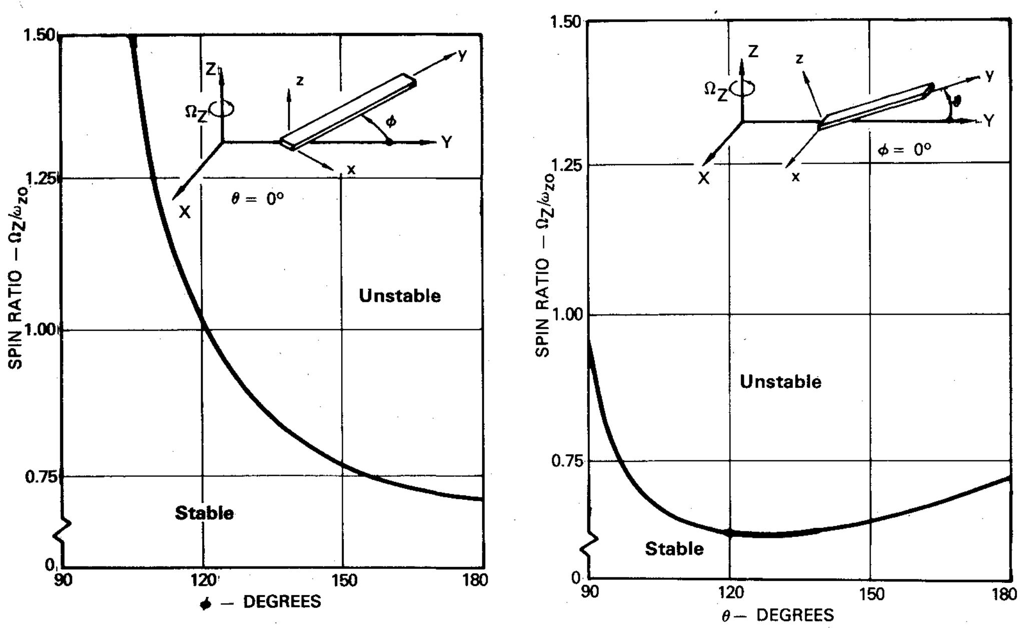

Fig. 7 Stability boundaries.

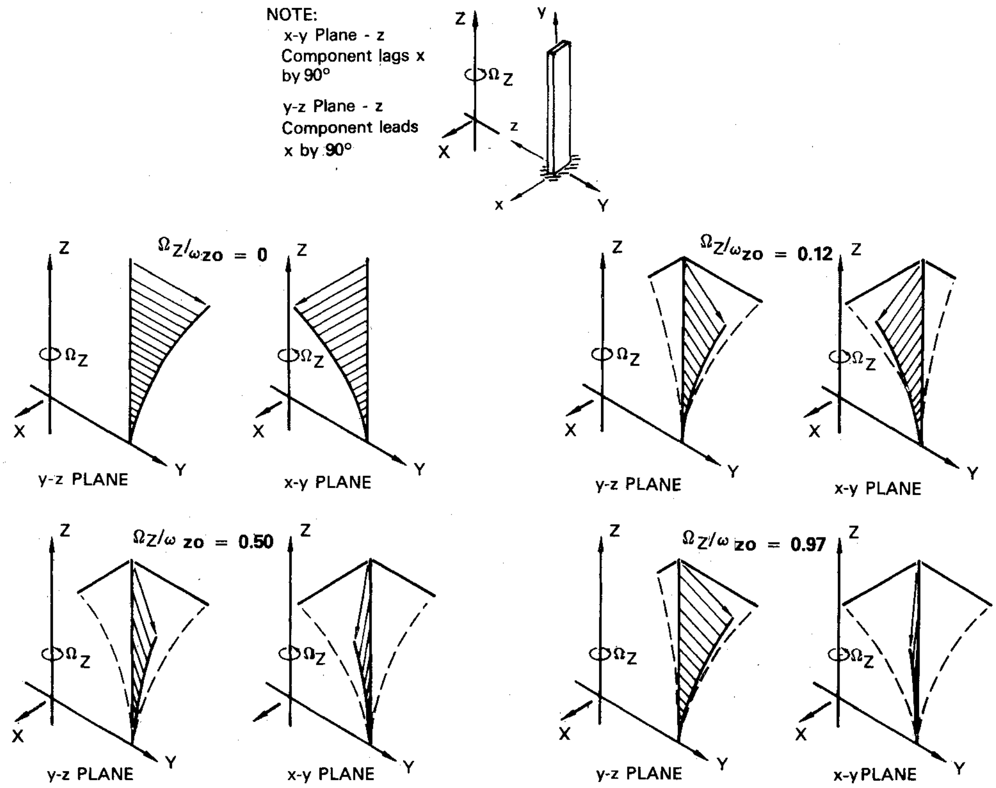

Fig. 8 Influence of spin on mode shapes.

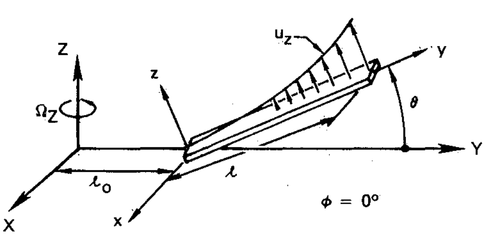

Fig. 9 Beam deformation in $y-z$ plane.

Employing the preceding definitions for velocity and elastic deformation, expressions are obtained for the kinetic and potential energies of the spinning beam. The total kinetic energy of the beam is given by

$$T=\frac{1}{2}\rho A\int_0^{\ell}(\dot{r}_x{}^2+\dot{r}_y{}^2+\dot{r}_z{}^2)\,\mathrm{d}y \tag{21}$$

where ρ is the beam mass density and A is its cross-sectional area. The total potential energy is made up of the strain energy V of bending and the potential V_C due to the centrifugal force field. The bending strain energy is given by

$$V=\frac{1}{2}\int_0^{\ell}\left[EI_x\left(\frac{d^2u_x}{dy^2}\right)^2+EI_z\left(\frac{d^2u_z}{dy^2}\right)^2\right]dy \tag{22}$$

where E is the modulus of elasticity of the beam material and I_x and I_z are the moments of inertia of the beam. Following the approach detailed in Ref. 1, the potential due to the centrifugal force field is expressed as

$$V_C=\frac{1}{2}\int_0^{\ell}\int_0^{y}\left[\left(\frac{du_x}{d\xi}\right)^2+\left(\frac{du_z}{d\xi}\right)^2\right]d\xi dF_C \tag{23}$$

where dF_C is the differential centrifugal force along the beam y axis and ξ is a dummy variable measured along y. The differential force quantity is defined as

$$dF_C=\rho A\Omega_Z^2[\cos\theta\cos\phi\ \ell_0 + (\sin^2\phi+\cos^2\theta\cos^2\phi)y]\,dy \tag{24}$$

It is of interest to note that there has been some recent discussions[13-14] as to alternate methods for incorporating the influence of the centrifugal force field in the analysis.

Combining the relationship of Eqs. (18), (19), (20), and (24) with the appropriate energy expressions [Eqs. (21), (22), and (23)] and performing the indicated operations leads to the final form for the energy equations. Due to the complex nature of these energy expressions their details are not presented here.

To obtain the governing equation of motion for the rth generalized coordinate, either q_x or q_z, the Lagrange Equation of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}+\frac{\partial}{\partial q_r}(V+V_C)=0 \tag{25}$$

is used. After performing the operations indicated by Eq. (25), the beam equations of motion take the form

$$M\ddot{q}+D\dot{q}+(K+K_G+K_C)U=0 \tag{26}$$

where q is the column matrix of generalized coordinates q_x and q_z. The coefficient matrices of equation (26) include the mass matrix M expressed as

$$M=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{27}$$

the contribution due to Coriolis acceleration

$$D=\begin{bmatrix}0 & 2\Omega_Z\sin\theta\cos\phi\\ -2\Omega_Z\sin\theta\cos\phi & 0\end{bmatrix} \tag{28}$$

the beam stiffness matrix K expressed as

$$K=\begin{bmatrix} \omega_{x0}^2 & 0 \\ 0 & \omega_{z0}^2 \end{bmatrix} \tag{29}$$

the contribution due to centripetal acceleration

$$K_C=\Omega_Z^2\begin{bmatrix} -\cos^2\theta\,\sin^2\phi-\cos^2\phi & \cos\theta\,\sin\theta\,\sin\phi \\ \cos\theta\,\sin\theta\,\sin\phi & -\sin^2\theta \end{bmatrix} \tag{30}$$

and the terms due to the steady-state centrifugal force field

$$K_G=\Omega_Z^2\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{31}$$

In Eq. (31) the centrifugal force parameter λ is defined as

$$\lambda=1.59\cos\theta\,\cos\phi\,(\ell_0/\ell)+1.23(\sin^2\phi+\cos^2\theta\,\cos^2\phi) \tag{32}$$

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